5 Linear Algebra

Solve the simultaneous equations $\mathbf{Ax} = \mathbf{b}$

5.1 Introduction

- A system of algebraic equations has the form

$$
\begin{aligned}
A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= b_1 \\
A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= b_2 \\
A_{31}x_1 + A_{32}x_2 + \cdots + A_{3n}x_n &= b_3 \\
&\vdots \\
A_{n1}x_1 + A_{n2}x_2 + \cdots + A_{nn}x_n &= b_n
\end{aligned}
$$

where the coefficients A_{ij} and the constants b_j are known, and x_i represent the unknowns.

$$
\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

$$
\left[\mathbf{A} \,|\, \mathbf{b}\right] = \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{n3} & b_n \end{array}\right]
$$

Matrix Form

Augmented coefficient matrix

$$\mathbf{Ax} = \mathbf{b}$$

5.1 Introduction: Uniqueness of Solution

- A system of *n* linear equations in *n* unknowns has a unique solution, provided that the determinant of the coefficient matrix is *nonsingular*, that is, if $|\mathbf{A}| = 0$.

- The rows and columns of a nonsingular matrix are *linearly independent* in the sense that no row (or column) is a linear combination of other rows (or columns).

- If the coefficient matrix is *singular*, the equations may have an infinite number of solutions, or no solutions at all, depending on the constant vector.

$$2x + y = 3 \qquad 4x + 2y = 6$$

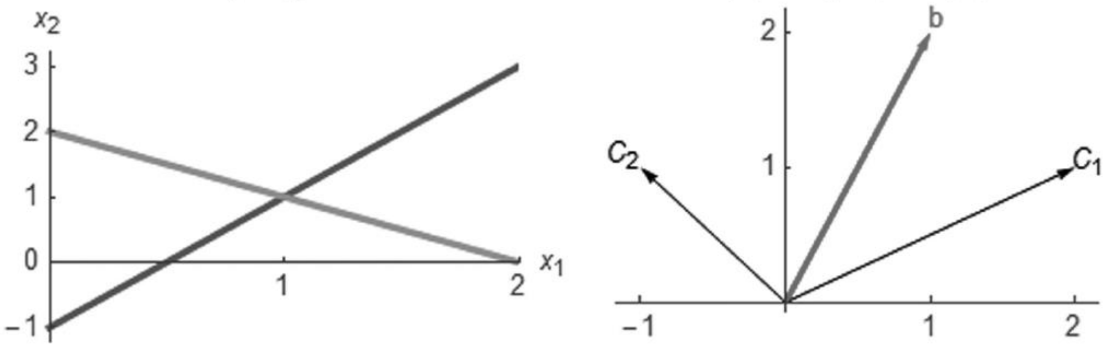
$$2x + y = 3 \qquad 4x + 2y = 0$$

5.1 Introduction: Uniqueness of Solution

Well behaved set of simultaneous equations with a unique solution

$$2x_1 - x_2 = 1$$
$$x_1 + x_2 = 2$$

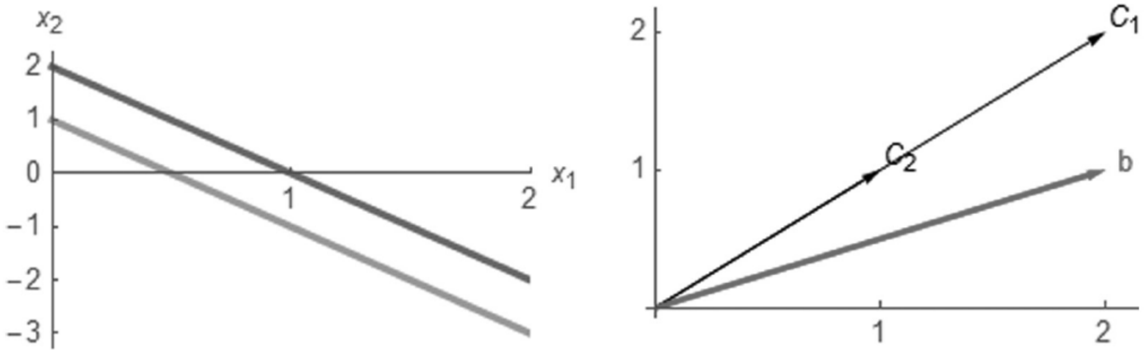
$$\begin{pmatrix}2\\1\end{pmatrix}x_1 + \begin{pmatrix}-1\\1\end{pmatrix}x_2 = \begin{pmatrix}1\\2\end{pmatrix}$$



A set of simultaneous equations with no solution.

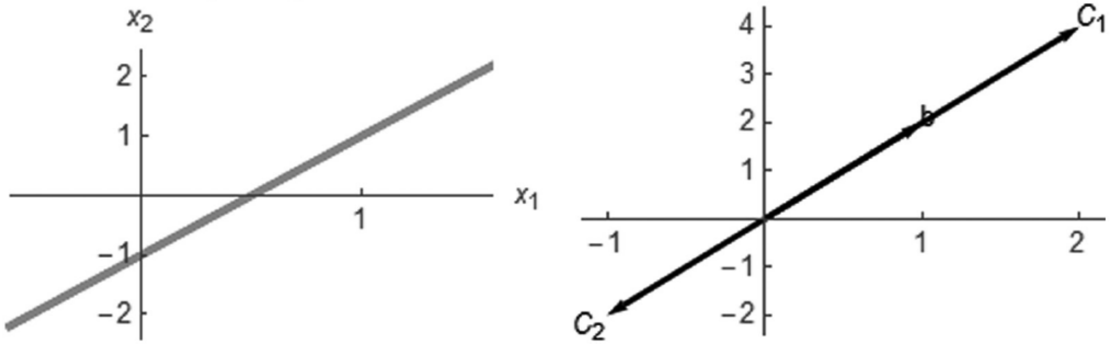
$$2x_1 + x_2 = 2$$
$$2x_1 + x_2 = 1$$

$$\begin{pmatrix}2\\2\end{pmatrix}x_1 + \begin{pmatrix}1\\1\end{pmatrix}x_2 = \begin{pmatrix}2\\1\end{pmatrix}$$



$$2x_1 - x_2 = 1$$
$$4x_1 - 2x_2 = 2$$

$$\begin{pmatrix}2\\4\end{pmatrix}x_1 + \begin{pmatrix}-1\\-2\end{pmatrix}x_2 = \begin{pmatrix}1\\2\end{pmatrix}$$



A set of simultaneous equations with an infinite number of solutions.

5.10 Solution Techniques: Direct Methods

Method	Initial form	Final form
Gauss elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ux} = \mathbf{c}$
LU decomposition	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{LUx} = \mathbf{b}$
Gauss–Jordan elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ix} = \mathbf{c}$

- In the above table,**U** represents an upper triangularmatrix, **L** is a lower triangular matrix, and **I** denotes the identity matrix.

- A square matrix is called *triangular*, if it contains only zero elements on one side of the principal diagonal.

$$\mathbf{U} = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

5.10 Solution Techniques: Gauss Elimination/Gauss Jordan Methods

Method	Initial form	Final form
Gauss elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ux} = \mathbf{c}$
Gauss–Jordan elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ix} = \mathbf{c}$

- In the above table,**U** represents an upper triangularmatrix, **L** is a lower triangular matrix, and **I** denotes the identity matrix.

- A square matrix is called *triangular*, if it contains only zero elements on one side of the principal diagonal.

- The Gaussian elimination algorithm consists of two basic steps: (1) eliminate the elements below the diagonal and (2) back substitute to get the solution.

5.10 Solution Techniques: Gauss Elimination Method

Method	Initial form	Final form
Gauss elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ux} = \mathbf{c}$
Gauss–Jordan elimination	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{Ix} = \mathbf{c}$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a'_{22} & a'_{23} \\ 0 & 0 & a''_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \end{bmatrix} \Rightarrow$$

$$x_3 = \frac{b''_3}{a''_{33}}$$

$$x_2 = \frac{b'_2 - a'_{23}x_3}{a'_{22}}$$

$$x_1 = \frac{b_1 - a_{12}x_2 - a_{13}x_3}{a_{11}}$$

- The Gauss–Jordan method is essentially Gauss elimination taken to its limit. In the Gauss elimination method only the equations that lie below the pivot equation are transformed.

- The main disadvantage of Gauss–Jordan elimination is that it involves about $n^3/2$ long operations, which is 1.5 times the number required in Gauss elimination.

5.10 Solution Techniques: Gauss Elimination Method

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=4,-2,1,11:-2,4,-2,-16:1,-2,4,17&ops=m0:1.4,r0:2:1,r0:-1:2,m1:1.3,r1:3.2:2,m2:1.3,r2:1.2:1,r2:-1.4:0,r1:1.2:0&cur=9

$$4x_1 - 2x_2 + x_3 = 11$$

$$-2x_1 + 4x_2 - 2x_3 = -16$$

$$x_1 - 2x_2 + 4x_3 = 17$$

$$\begin{bmatrix} \color{red}{4} & -2 & 1 & 11 \\ \color{red}{-2} & 4 & -2 & -16 \\ \color{red}{1} & -2 & 4 & 17 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 1 & 11 \\ -2 & 4 & -2 & -16 \\ 1 & -2 & 4 & 17 \end{bmatrix} \xrightarrow{R_1 = \frac{1}{4}R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ -2 & 4 & -2 & -16 \\ 1 & -2 & 4 & 17 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 + 2R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 3 & -\frac{3}{2} & -\frac{21}{2} \\ 1 & -2 & 4 & 17 \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 - 1R_1} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 3 & -\frac{3}{2} & -\frac{21}{2} \\ 0 & -\frac{3}{2} & \frac{15}{4} & \frac{57}{4} \end{bmatrix}$$

$$\xrightarrow{R_2 = \frac{1}{3}R_2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & -\frac{3}{2} & \frac{15}{4} & \frac{57}{4} \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + \frac{3}{2}R_2} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 3 & 9 \end{bmatrix}$$

$$\xrightarrow{R_3 = \frac{1}{3}R_3} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_2 = R_2 + \frac{1}{2}R_3} \begin{bmatrix} 1 & -\frac{1}{2} & \frac{1}{4} & \frac{11}{4} \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - \frac{1}{4}R_3} \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{R_1 = R_1 + \frac{1}{2}R_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

5.10 Solution Techniques: Gauss Elimination Method

$$\begin{bmatrix} 0 & 1 & 2 & 5 \\ 5 & 3 & 2 & -3 \\ 2 & -1 & 6 & 4 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{bmatrix} 5 & 3 & 2 & -3 \\ 0 & 1 & 2 & 5 \\ 2 & -1 & 6 & 4 \end{bmatrix}$$

$$\xrightarrow{R_1 = \frac{1}{5}R_1} \begin{bmatrix} 1 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 2 & 5 \\ 2 & -1 & 6 & 4 \end{bmatrix} \xrightarrow{R_3 = R_3 - 2R_1} \begin{bmatrix} 1 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 2 & 5 \\ 0 & -\frac{11}{5} & \frac{26}{5} & \frac{26}{5} \end{bmatrix}$$

$$\xrightarrow{R_3 = R_3 + \frac{11}{5}R_2} \begin{bmatrix} 1 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 2 & 5 \\ 0 & 0 & \frac{48}{5} & \frac{81}{5} \end{bmatrix} \xrightarrow{R_3 = \frac{5}{48}R_3} \begin{bmatrix} 1 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 1 & \frac{27}{16} \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_3} \begin{bmatrix} 1 & \frac{3}{5} & \frac{2}{5} & -\frac{3}{5} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{27}{16} \end{bmatrix} \xrightarrow{R_1 = R_1 - \frac{2}{5}R_3} \begin{bmatrix} 1 & \frac{3}{5} & 0 & -\frac{51}{40} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{27}{16} \end{bmatrix}$$

$$\xrightarrow{R_1 = R_1 - \frac{3}{5}R_2} \begin{bmatrix} 1 & 0 & 0 & -\frac{9}{4} \\ 0 & 1 & 0 & \frac{13}{8} \\ 0 & 0 & 1 & \frac{27}{16} \end{bmatrix}$$

https://textbooks.math.gatech.edu/ila/demos/rrinter.html?mat=0,1,2,5:5,3,2,-3:2,-1,6,4&ops=s1:0,m0:1.5,r0:-2:2,r1:11.5:2,m2:5.48,r2:-2:1,r2:-2.5:0,r1:-3.5:0&cur=8

5.11 Solution Techniques: LU Factorization

Method	Initial form	Final form
LU decomposition	$\mathbf{Ax} = \mathbf{b}$	$\mathbf{LUx} = \mathbf{b}$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}, \quad \mathbf{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

The LU decomposition algorithm is

- Decompose or factor **A** into **LU**.
- Use forward substitution to solve $\mathbf{L} \cdot \mathbf{d} = \mathbf{b}$ for **d**.
- Use back substitution to solve $\mathbf{U} \cdot \mathbf{x} = \mathbf{d}$ for **x**.

5.11 Solution Techniques: LU Factorization

The LU decomposition algorithm is

- Decompose or factor **A** into **LU**.
- Use forward substitution to solve **L**·**d** = **b** for **d**.
- Use back substitution to solve **U**·**x** = **d** for **x**.

$$4x_1 - 2x_2 + x_3 = 11$$

$$-2x_1 + 4x_2 - 2x_3 = -16$$

$$x_1 - 2x_2 + 4x_3 = 17$$

$$\begin{bmatrix} 4 & -2 & 1 & 11 \\ -2 & 4 & -2 & -16 \\ 1 & -2 & 4 & 17 \end{bmatrix}$$

```
[L,U]=lu([4 -2 1;-2 4 -2;1 -2 4])

L =

    1.0000         0         0
   -0.5000    1.0000         0
    0.2500   -0.5000    1.0000


U =

    4.0000   -2.0000    1.0000
         0    3.0000   -1.5000
         0         0    3.0000
```

$$\begin{bmatrix} 1 & 0 & 0 \\ -0.5 & 1 & 0 \\ 0.25 & -0.5 & 1 \end{bmatrix} \begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ -16 \\ 17 \end{Bmatrix}$$

$$\begin{Bmatrix} d_1 \\ d_2 \\ d_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ -10.5 \\ 9 \end{Bmatrix}$$

$$\begin{bmatrix} 4 & -2 & 1 \\ 0 & 3 & -1.5 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 11 \\ -10.5 \\ 9 \end{Bmatrix}$$

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -2 \\ 3 \end{Bmatrix}$$

5.12 Solution Techniques: Iterative Methods (Gauss-Seidel)

$$\begin{bmatrix} 4 & -1 & 1 \\ -1 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -1 \\ 5 \end{bmatrix}$$

$$x_1 = \frac{1}{4}(12 + x_2 - x_3)$$

$$x_2 = \frac{1}{4}(-1 + x_1 + 2x_3)$$

$$x_3 = \frac{1}{4}(5 - x_1 + 2x_2)$$

Choosing the starting values $x_1 = x_2 = x_3 = 0$, the first iteration gives us

$$x_1 = \frac{1}{4}(12 + 0 - 0) = 3$$

$$x_2 = \frac{1}{4}[-1 + 3 + 2(0)] = 0.5$$

$$x_3 = \frac{1}{4}[5 - 3 + 2(0.5)] = 0.75$$

5.12 Solution Techniques: Iterative Methods (Gauss-Seidel)

The second iteration yields

$$x_1 = \frac{1}{4}(12 + 0.5 - 0.75) = 2.9375$$

$$x_2 = \frac{1}{4}[-1 + 2.9375 + 2(0.75)] = 0.859\,38$$

$$x_3 = \frac{1}{4}[5 - 2.9375 + 2(0.85938)] = 0.945\,31$$

and the third iteration results in

$$x_1 = \frac{1}{4}(12 + 0.85938 - 0.94531) = 2.978\,52$$

$$x_2 = \frac{1}{4}[-1 + 2.97852 + 2(0.94531)] = 0.967\,29$$

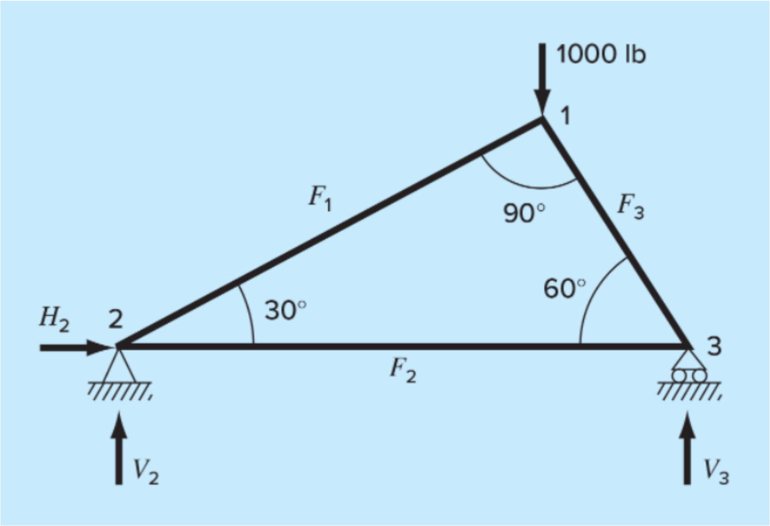
$$x_3 = \frac{1}{4}[5 - 2.97852 + 2(0.96729)] = 0.989\,02$$

After five more iterations the results would agree with the exact solution $x_1 = 3$, $x_2 = x_3 = 1$ within five decimal places.

5.13 Exercise (Solve using Gauss Elimination, LU Decomposition, Gauss-Jordan and Gauss-Seidel)

$$
\begin{aligned}
5x_1 - 2x_2 + 3x_3 &= -1 \\
-3x_1 + 9x_2 + x_3 &= 2 \\
2x_1 - x_2 - 7x_3 &= 3
\end{aligned}
$$

5.13 Case Study



$$F_1 = -500 \qquad F_2 = 433 \qquad F_3 = -866$$
$$H_2 = 0 \qquad V_2 = 250 \qquad V_3 = 750$$

$$\begin{bmatrix} 0.866 & 0 & -0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.866 & 0 & 0 & 0 \\ -0.866 & -1 & 0 & -1 & 0 & 0 \\ -0.5 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.866 & 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ H_2 \\ V_2 \\ V_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

References

- *Applied Engineering Mathematics*, Brian Vick, CRC Press, 2020
- *Numerical Methods in Engineering with MATLAB*, Jaan Klusalaas, Cambridge University Press, 2012